Series expansion analysis of the backbone properties of two-dimensional percolation clusters

1997 J. Phys. A: Math. Gen. 30 6215

(http://iopscience.iop.org/0305-4470/30/18/008)

J. Phys. A: Math. Gen. **30** (1997) 6215–6232. Printed in the UK PII: S0305-4470(97)80616-6

Series expansion analysis of the backbone properties of two-dimensional percolation clusters

F M Bhatti†, R Brak‡, J W Essam§ and T Lookman‖

† Department of Mathematics, University of Brunei Darussalam, BSB 2028, Brunei Darussalam
‡ Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia
§ Department of Mathematics, Royal Holloway, University of London, Egham Hill, Egham, Surrey TW20 0EX, UK
‖ Department of Applied Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B9

Received 30 December 1996, in final form 23 June 1997

Abstract. Low-density series expansions for the backbone properties of two-dimensional bond percolation clusters are derived and analysed. Expansions for most of the 14 properties considered are new and are obtained to order p^{18} on the square lattice and order p^{14} on the triangular lattice. Earlier series work was confined to three properties of the square lattice and was to order p^{10}. The fractal dimension of the bonds or sites in the backbone is estimated to be $D_{\mathrm{B}} = 1.605 \pm 0.015$ and is intermediate between a previously conjectured field theory value and the latest Monte Carlo results. The union, intersection and length of the longest self-avoiding paths are found to have the same fractal dimension which is close to D_{B} and consistent with the field theory conjecture for D_{B}. On the other hand, the union intersection and length of the shortest paths are found to have different dimensions and in the case of the intersection, the triangular and square lattices are found to have sigificantly different dimensions. The fractal dimension of the shortest path also appears to be non-universal and we find $d_{\mathrm{min}} = 1.106 \pm 0.007$ for the square lattice and 1.148 ± 0.007 for the triangular lattice. Critical amplitude ratios are considered and found to be in agreement with theoretical inequalities.

1. Introduction

The backbone of the infinite cluster above the percolation threshold p_c was considered by Skal and Shklovskii [1] and de Gennes [2] in their theoretical work on the critical behaviour of the conductivity of random resistor networks. The backbone was defined as that part of the cluster which can carry current when electrodes are placed across opposite faces of a rectangular sample. Later, Pike and Stanley [3] focused their attention on the geometry of the backbone and used Monte Carlo methods to estimate its fractal dimension, D_{B}, and that of the cutting bonds. Here we study the fractal geometry of percolation clusters by series expansion methods.

Harris and Fisch [4] showed that series expansions in powers of p, the probability that a given bond is conducting, may be obtained by considering only finite clusters. They studied the resistive susceptibility $\chi_{\mathrm{R}}(p)$ and showed that the exponent γ_{R} with which it diverges as p approaches p_c from below also determines the way in which the conductivity of the infinite cluster approches zero from above p_c. Later, Hong and Stanley [5] obtained similar expansions for the geometrical properties of the backbone, the exponents of which determine the fractal dimensions.

For any configuration and pair of lattice sites $\{u, v\}$, the $u - v$ backbone, b_{uv}, may be defined as the two-rooted graph formed by taking the union of all paths of conducting bonds connecting u and v. If u and v are not connected then b_{uv} is the null graph. A $u - v$ backbone variable Z_{uv} is a random variable whose value in any configuration depends only on b_{uv}. The corresponding 'susceptiblity' $\chi_Z(p)$ is defined in a similar manner to the resistive susceptibility [4] by

$$\chi_Z(p) = \sum_v \mathcal{E}(Z_{uv}) \tag{1}$$

where the expected value $\mathcal{E}$ is taken over all configurations of conducting bonds and is independent of u since all sites are assumed to be equivalent.

The Z_{uv} we consider are the numbers of bonds or sites in various subsets of the backbone. Thus $\chi_Z(p)$ will diverge as p approaches p_c from below and we denote the corresponding dominant critical exponent by γ_Z. Another divergent function is the expected number of sites, $S(p)$, which are connected to u by a path of conducting bonds and has critical exponent γ. The average value of Z_{uv} may be estimated by normalizing the sum in (1) by dividing by $S(p)$ and the resulting function has critical exponent $\zeta_Z = \gamma_Z - \gamma$. If Z_{uv} is the size of some subset of bonds or sites the fractal dimension of the subset is given by $d_Z = \zeta_Z/\nu$ where ν is the critical exponent of the connectedness length. For example, if Z_{uv} is the number of bonds in the whole backbone b_{uv} then $d_Z = D_{\mathrm{B}}$, the fractal dimension referred to above.

In this paper we analyse the series expansions corresponding to 14 different Z_{uv} variables. The first four of these are the numbers of bonds or sites in either the union or intersection of all paths connecting u and v and the corresponding ζ_Z exponents will be denoted by ζ_{BU}, ζ_{SU}, ζ_{BI} and ζ_{SI}. Note that $\zeta_{\mathrm{BU}} = \nu D_{\mathrm{B}}$ since the union of paths gives the whole backbone. The bonds which lie in the intersection are also known as nodal or cutting bonds and Coniglio has shown [6] that $\zeta_{\mathrm{BI}} = 1$. A further four functions may be defined by considering only the shortest paths connecting u and v. The union of these paths has been called the elastic backbone [7] and its fractal dimension denoted by D_{E}. A certain duality has been shown to exist between shortest and longest paths [8] and we also consider the susceptibilities arising from the numbers of bonds and sites in the union and intersection of the longest self-avoiding paths. The final two functions we consider are obtained by taking Z_{uv} to be the length of either the shortest path or the longest self-avoiding path in the backbone the exponents of which are denoted by $\zeta_{\min}$ and $\zeta_{\max}$. The first of these exponents is related to the spreading dimension $\hat{d}$ [9, 10], by $\hat{d} = \Delta/\zeta_{\min}$ where in two dimensions the gap exponent Δ has the value [11] $\frac{91}{36}$.

The exponents defined above clearly satisfy the following constraints, some of which were given by Coniglio [12]. The exponents for the whole backbone satisfy

$$\nu \leqslant \zeta_{\min} \leqslant \zeta_{\max} \leqslant \zeta_{\mathrm{SU}} = \zeta_{\mathrm{BU}}. \tag{2}$$

For the shortest paths

$$\zeta_{\mathrm{BI}} = \zeta_{\mathrm{SI}} \leqslant \zeta_{\min} \leqslant \zeta_{\mathrm{SU}} = \zeta_{\mathrm{BU}} \tag{3}$$

and for the longest paths

$$\zeta_{\mathrm{BI}} = \zeta_{\mathrm{SI}} \leqslant \zeta_{\max} \leqslant \zeta_{\mathrm{SU}} = \zeta_{\mathrm{BU}}. \tag{4}$$

The equalities for bond and site unions follow from the fact that the union of paths is a connected subgraph of the lattice and if such a graph has s sites and b bonds then

$$s - 1 \leqslant b < \tfrac{1}{2}zs \tag{5}$$

where z is the lattice coordination number. Similarly, the bond and site intersections are subsets of the elements of a single chain and the equality of the intersection exponents follows from the relation

$$b + 1 \leqslant s \leqslant 2b. \tag{6}$$

Hong and Stanley [5] obtained series expansions for $\chi_{\mathrm{BU}}(p)$, $\chi_{\min}(p)$ and $\chi_{\mathrm{BI}}(p)$ to order p^{10} on a general hypercubic lattice. Here the susceptibilities are those corresponding to all paths. By restricting our attention to the square lattice we have extended their work by a further eight terms and obtained agreement with all but their last term. More recently, Adler *et al* [13] derived an expansion to order p^{13} for a function having the same critical exponent as $\chi_{\mathrm{BU}}(p)$. This arose in their study of the moments of the current distribution and is the zeroth moment of that distribution. It is in fact equal to the number of bonds in the path union which actually carry a non-zero current. Some bonds in the union carry zero current due to symmetry, for example this will occur for a balanced Wheatstone bridge. The smallest example of this on the hypercubic lattice occurs at order 7 which accounts for the fact that only the first six terms of [13] agree with Hong and Stanley [5] who used the definition given here. In order to obtain further independent estimates of the two-dimensional critical exponents we have also generated the first 14 terms for the same functions on the triangular lattice.

Recent Monte Carlo work on the fractal dimension D_{B} has produced apparently very accurate results which disagree with the rational value $D_{\mathrm{B}} = 1\frac{9}{16} = 1.5625$ allowed by conformal field theory which was considered as a possibility by Larsson [14] and conjectured as exact by Saleur [15]. Using high-statistics simulations of bond percolation on the square lattice, Grassberger [16] found $D_{\mathrm{B}} = 1.647 \pm 0.004$ and independently Rintoul and Nakanishi [17] obtained $D_{\mathrm{B}} = 1.64 \pm 0.01$. The large majority of our estimates favour the intermediate value $D_{\mathrm{B}} = 1.605 \pm 0.015$ but it is possible to find series and methods of analysis which agree with either the Monte Carlo or field theory values. Earlier Monte Carlo work gave 1.62 ± 0.02 for bond percolation on the square lattice [18] and for the triangular lattice, 1.61 ± 0.01 for site percolation [19] and 1.62 ± 0.06 for bond percolation [20]. The series analysis of Adler *et al* [13], using their 13 term expansion of the zeroth moment of the current distribution, gave $D_{\mathrm{B}} = 1.55 \pm 0.06$ in agreement with the field theory. However, they comment that a higher value would be found by adjusting the correction to scaling exponent to agree with that obtained from the mean size expansion.

Our results for the intersection of all paths confirm Coniglio's result [6]

$$\chi_{\mathrm{BI}}(p) = S'(p) \tag{7}$$

term by term in the expansion, which leads to $\zeta_{\mathrm{BI}} = 1$, and our analysis of the $\chi_{\mathrm{SI}}(p)$ expansions is consistent with $\zeta_{\mathrm{SI}} = 1$.

Table 1 shows the ζ-exponents corresponding to the shortest path susceptibilities.

The results are consistent with the equalities in (3) and we conclude that the inner inequalities are strict ($\zeta_{\mathrm{SI}} < \zeta_{\min} < \zeta_{\mathrm{SU}}$). By comparison, Herrmann *et al* [7] found from Monte Carlo on the square lattice that $\zeta_{\mathrm{SI}} = \zeta_{\min} = \zeta_{\mathrm{SU}} = \nu D_{\mathrm{E}}$ with $D_{\mathrm{E}} = 1.10 \pm 0.05$.

Table 1. Estimates of the ζ-exponents from the 'shortest path' series.

	ζ_{BI}	ζ_{SI}	$\zeta_{\min}$	ζ_{SU}	ζ_{BU}
Square	1.34 ± 0.03	1.38 ± 0.03	1.475 ± 0.010	1.58 ± 0.02	1.58 ± 0.02
Triangular	1.47 ± 0.01	1.47 ± 0.01	1.53 ± 0.01	1.57 ± 0.02	1.58 ± 0.01

Our exponents for path unions are the same for square and triangular lattices and give a fractal dimension for the elastic backbone $D_{\mathrm{E}} = 1.185 \pm 0.015$. However, it can be seen from table 1 that the exponents for the path length and path intersection appear to depend on the lattice which raises the question of possible non-universality for these functions. The difference for the intersection is well outside our estimated errors. Converting to fractal dimensions we find $d_{\min} = 1.106 \pm 0.007$ on the square lattice and $d_{\min} = 1.148 \pm 0.007$ for the triangular lattice. A difference in the same direction exists in the more recent Monte Carlo values 1.130 ± 0.002 [9], 1.1307 ± 0.0004 [16] for the square lattice and 1.15 ± 0.02 [19] for the triangular lattice. Earlier estimates of $\zeta_{\min}$ are summarized in [10] where it is denoted by $\nu_{\|}$.

Note that ζ for the path intersection on the square lattice is close to $\nu = \frac{4}{3}$ which means that this subset is essentially one-dimensional whereas for the triangular lattice it has fractal dimension 1.10 ± 0.01 similar to that for the shortest path length on the square lattice.

In the case of the longest paths our analysis strongly suggests that all of the exponents in (4) are equal, indicating that there is typically only one longest self-avoiding path between two connected sites in any given configuration. The values for the triangular lattice are better converged and are all consistent with a common exponent $\zeta_{\max} = 2.08 \pm 0.03$. Some of our estimates for the square lattice lie below this range but the results are generally less well converged and there is no strong evidence of non-universality. It is interesting to note that $\zeta_{\max} = \frac{25}{12}$ is a possible rational value which would be equal to νD_{B} using the field theory value [14, 15] of D_{B} and $\nu = \frac{4}{3}$. From (2) it follows that $\zeta_{\max} \leqslant \nu D_{\mathrm{B}}$ and equality would hold if the longest self-avoiding path were typically close to Hamiltonian.

2. Derivation of the series expansions

The methods used in deriving the series expansions have been described previously [8, 21]. To order 16 on the square lattice we used both the 'weight factor' method and the 'extended perimeter method' to provide a check on the programming. However, when many different functions are being expanded the latter method is more efficient and this was used to extend the square and derive the triangular lattice series. The methods will now be summarized.

A set of self-avoiding paths connecting the root points of a two-rooted graph will be said to *cover* the graph if every edge belongs to at least one of the paths. A two-rooted graph is a backbone if the set of all self-avoiding paths covers the graph. To obtain the expansions to order p^n both methods require the compilation of a list of non-isomorphic subgraphs $\{g_1, g_2, \ldots\}$ of the lattice which can be made into backbones by choosing two of the vertices to be root points. Let $\mathcal{B}(g_i)$ be the set of all possible backbones obtained by assigning the root labels u and v to two of the vertices of g_i. For example if g_i is a chain of any length then just two backbones can be formed by rooting the terminal vertices. Let the list be partially ordered so that if $i < j$ then $g_i \subset g_j$. The lattice constant L_i of g_i is the number of inequivalent ways in which g_i can occur as a subgraph of the lattice where two subgraphs are equivalent if they differ only by a translation. The number of edges in g_i will be denoted by ϵ_i.

The weight factor method is the simplest to describe and is based on the formula

$$\chi_Z(p) = \sum_{i=1}^{\infty} W_Z(g_i) L_i p^{\epsilon_i} \tag{8}$$

where the weight $W_Z(g_i)$ depends on Z and g_i but is independent of the lattice. It may be

Table 2. Correspondence between Z and Z^*.

Z	U	X	$U^{\min}$	$U^{\max}$	$X^{\min}$	$X^{\max}$	$L^{\min}$	$L^{\max}$
Z^*	X	U	$X^{\max}$	$X^{\min}$	$U^{\max}$	$U^{\min}$	$L^{\max}$	$L^{\min}$

written

$$W_Z(g_i) = \sum_{b\in\mathcal{B}(g_i)} w(b). \tag{9}$$

The partial weight $w(b)$ of the backbone b is given by

$$w(b) = \sum_{K\subseteq P:K \text{ covers } b} (-1)^{|K|+1} Z_K^* \tag{10}$$

where P is the set of all self-avoiding paths connecting the roots of b. It is shown in [8] that Z^* is in some sense dual to Z. Thus if $Z = X$, the number of elements (bonds or sites) in the intersection of P, then $Z_K^* = U_K$, the number of elements in the union of the paths K. Further, if $Z = X^{\min}$, the number of elements in the intersection of the shortest paths of P, then $Z_K^* = U_K^{\max}$, the number of elements in the union of the longest paths in K. Table 2 lists all of the required correspondences. $L^{\min}$ and $L^{\max}$ are respectively the lengths of the shortest and longest paths.

The extended perimeter method [21] is essentially a rearrangement of the weight factor method in which the weight is calculated directly in terms of Z and is therefore easily changed to obtain the expansion of a different function. The penalty for this is that the lattice information $L_i p^{\epsilon_i}$ has to be replaced by an infinite series $\theta_i(p)$ which is truncated at order p^n. The method uses the following equations

$$\chi_Z(p) = \sum_{i=1}^{\infty} Y(g_i)\theta_i(p) \tag{11}$$

with

$$Y(g_i) = \sum_{b\in\mathcal{B}(g_i)} Z(b). \tag{12}$$

The θ functions are calculated by solving the following equations recursively

$$\sum_{j=1}^{m} B_{ij}\theta_j(p) = L_i p^{\epsilon_i} \tag{13}$$

where m is the number of graphs in the list with $\leqslant n$ edges and B_{ij} is the number of subgraphs of g_j which are isomorphic to g_i. The matrix B is upper triangular as a result of the assumed ordering of the graphs. The θ function of a given graph g_i is obtained by starting with $L_i p^{\epsilon_i}$ and subtracting a linear combination of the θ functions of its supergraphs which are in the list. Thus, if the series are required to order p^n then, correct to this order, graphs with n edges have $\theta_i(p) = L_i p^n$. Graphs with $n - r$ edges have $\theta_i(p) = p^{n-r}\phi_i(p)$ where $\phi_i(p)$ is a polynomial of degree i. The extended perimeter method is demonstrated to order p^7 in [21].

Both methods require generation of the graph list and corresponding lattice constants which was done in the following stages.

(a) A list G of lattice subgraphs is made having no articulation points and $\leqslant n$ edges.

(b) Two further lists R_1 and R_2 are made by rooting the graphs of G with one or two roots respectively in all possible inequivalent ways.

(c) A list of nodal graphs N is made by stringing together the graphs in R_1 and R_2. A nodal graph with k non-nodal parts $\{h_1, h_2, \ldots, h_k\}$ is obtained by choosing h_1 and h_k from the list R_1 and any intermediate non-nodal parts are chosen from R_2. The graphs are then joined at their root points in all possible ways taking account of symmetry to avoid duplicates.

(d) The lists G and N are merged in the required partial order and stored on disk for further processing.

(e) The lattice constant is calculated for each graph and stored along with its description.

In the weight factor method the Z^* functions for the path subsets are computed and hence the weight W is calculated for each graph in the list. The series are then formed using (8).

In the extended perimeter method the θ functions are next calculated and then the weights Y in terms of Z. The calculation of Y is relatively fast since path subsets are not required. The series are then formed using (11).

The resulting expansions for the triangular and square lattices are given in the appendix. In calculating the size of the site unions and intersections we have not counted the intial site u and path length is measured in terms of bonds rather than sites. These conventions make the constant term in all of the χ expansions equal to zero. Since we are considering bond percolation the site u is always present and the effect of counting this site would be to make the first term of the χ expansion equal to 1 and hence the critical exponent would be unchanged. In estimating the critical exponents we have considered the expansions with the first term equal to both 0 and 1.

The correctness of our graph generation code was checked by comparison with the results of Conway and Guttmann [22] who generated the first 18 terms of the mean size by bonds on the square lattice by an independent method.

3. Analysis of the expansions

The expansions were analysed on the assumption that as p approaches p_c from below

$$\chi_Z(p) \cong A_Z(1 - p/p_c)^{-\gamma_Z}(1 + a(1 - p/p_c)^{\Delta_1} + \text{higher-order terms}) \qquad (14)$$

where γ_Z is the critical exponent and Δ_1 is the leading correction to scaling exponent. The subscript indicates that Δ_1 is the first of a sequence $\Delta_1, \Delta_2, \ldots$ of correction exponents and distinguishes it from the gap exponent Δ of the cluster size distribution. The dependence of Δ_1 on Z has been supressed but we have no reason to believe that it is the same for all properties considered. This is a notational convenience and the relevant property will always be clear from the context. For the square lattice $p_c = \frac{1}{2}$ and for the triangular lattce $p_c = 2\sin(\pi/18)$.

For each of the 28 χ expansions we have obtained nine estimates of the corresponding critical exponent. Three Padé approximant methods which allow for corrections to scaling were used; the method of Adler *et al* [23] which was called M2 in [24], the M1 method [24] and the method of Baker and Hunter (BH) [25]. Each method was applied to the expansions of $1 + \chi(p)$, $\chi(p)/p$ and $\mathrm{d}\chi(p)/\mathrm{d}p$. The error assigned to each estimate is a measure of the consistency of the corresponding Padé approximants but the overall error in a given exponent is better measured by the spread in the different estimates (see also [26]).

The M2 method was nearly always satisfactory and gave the best convergence. The BH method frequently gave results which were difficult to analyse due to the occurrence of defects which disturbed the estimate of the dominant exponent and made a meaningful estimate of Δ_1 impossible. These defects occurred mainly in the higher-order approximants and in this case disappointingly little use could be made of the terms of the expansion which contained the most significant information.

The results of the exponent analysis are listed in tables 3, 5, 6, 8–10. In the tables an asterisk by the Baker–Hunter estimate means that no satisfactory estimate of Δ_1 could be obtained due to the widespread appearance of defective approximants. A dagger in an M2 row indicates that the γ versus Δ_1 distribution was poorly converged but rather flat so that γ_Z could be estimated but not Δ_1. A double dagger in an M1 or M2 row indicates that there was a strong variation of γ with Δ_1 with no obvious converged region so that neither exponent is estimated.

The critical amplitude A_Z was estimated by a method similar to M2. Assuming a value for γ_Z the series expansion of $\alpha_Z(p) = \chi_Z(p)(1-p/p_c)^{\gamma_Z}$ in powers of p was first formed. Corrections to scaling were taken into account by using the transformation

$$p = p_c(1-(1-y)^{\frac{1}{z}}). \tag{15}$$

In terms of y

$$\alpha_Z(p(y)) \cong A_Z(1+a(1-y)^{\Delta_1/z}). \tag{16}$$

Assuming a value of z, p was expanded as a series of powers of y and then substituted into the expansion of $\alpha_Z(p)$. Padé approximants were then made to the resulting series in y and evaluated at $y = 1$ to obtain an estimate of A_Z. A sequence of values of z in the range 0.5–3.5 was used and, from (16), the best convergence should be obtained when $z = \Delta_1$ since the first correction to scaling is then analytic as a function of y. In cases when two functions have the same critical point and exponent the ratio of their amplitudes may be estimated in a similar way by first forming the power expansion of the ratio of the functions and using it instead of $\alpha_Z(p)$.

3.1. Intersection and union of all paths

3.1.1. Intersection of all paths. Table 3 shows the estimates of ζ for the number of bonds and sites in the intersection of all paths (nodal bonds and sites in the backbone). The column headings emphasize that ζ_Z is obtained by estimating γ_Z and then subtracting the value $\gamma = 2\frac{7}{18}$ [11] which is normally accepted as being exact.

Coniglio has shown that $\zeta_{\rm BI} = 1$ and in the introduction we showed that $\zeta_{\rm SI} = \zeta_{\rm BI}$ so that our data is merely indicating the accuracy to be expected in the subsequent series analysis. Notice that the error bars on individual estimates are smaller than the general spread of values and that the exact value in many cases falls slightly outside the quoted ranges. For the triangular lattice the estimates of $\zeta_{\rm SI}$ are consistently a few per cent higher than the theoretical value. This will be relevant to the discussion of results in subsequent cases when exact values are not known. It must be borne in mind that the error bars are a subjective measure of the convergence of the Padé approximants and are not strict bounds. The general spread is therefore a more reliable indicator of the accuracy. The observations on the data of table 3 made in the following two paragraphs apply to all subsequent tables.

In methods M1 and M2 a range of Padé approximants is considered, each of which would determine the exact values of γ_Z and Δ_1 if the higher-order terms in (14) were not present. In practice a given approximant defines a curve in the $\gamma_Z - \Delta_1$ plane and with the aid of a graphics displays a region of the plane is then sought in which the majority of curves have coalesced. The quoted ranges of γ_Z and Δ_1 define the limits of this region. Notice that the range of Δ_1 is much wider than that of the leading exponent. More importantly, the estimates of Δ_1 for a given property and lattice depend quite strongly on the series and

Table 3. Estimates of the ζ-exponent for the intersection of all paths.

		Square		Triangular	
		$\gamma_{BI}-\gamma$	Δ_1	$\gamma_{BI}-\gamma$	Δ_1
(a) Bond Intersection					
	M2	1.01 ± 0.01	1.30 ± 0.07	0.99 ± 0.01	1.1 ± 0.1
$1+\chi_{BI}$	M1	1.04 ± 0.04	1.45 ± 0.15	0.99 ± 0.20	0.8 ± 0.2
	BH	1.00 ± 0.03	1.13 ± 0.12	0.97 ± 0.03	0.9 ± 0.2
	M2	0.98 ± 0.02	1.30 ± 0.09	0.99 ± 0.20	2.15 ± 0.15
χ_{BI}/p	M1	‡	‡	0.993 ± 0.004	2.2 ± 0.3
	BH	0.99 ± 0.03	1.09 ± 0.11	0.98 ± 0.01	2.2 ± 0.7
	M2	1.04 ± 0.04	1.55 ± 0.05	1.00 ± 0.02	1.1 ± 0.1
$d\chi_{BI}/dp$	M1	1.03 ± 0.02	1.5 ± 0.3	0.99 ± 0.01	0.9 ± 0.2
	BH	1.01 ± 0.03	1.2 ± 0.2	0.98 ± 0.02	0.94 ± 0.07
		$\gamma_{SI}-\gamma$	Δ_1	$\gamma_{SI}-\gamma$	Δ_1
(b) Site intersection					
	M2	‡	‡	1.05 ± 0.02	1.60 ± 0.08
$1+\chi_{SI}$	M1	‡	‡	1.03 ± 0.02	1.25 ± 0.05
	BH	1.01 ± 0.06	1.25 ± 0.12	1.03 ± 0.03	*
	M2	1.05 ± 0.02	1.6 ± 0.2	1.08 ± 0.03	1.25 ± 0.07
χ_{SI}/p	M1	‡	‡	‡	‡
	BH	0.98 ± 0.02	1.29 ± 0.13	1.06 ± 0.05	1.14 ± 0.09
	M2	1.04 ± 0.03	1.30 ± 0.05	1.03 ± 0.02	1.25 ± 0.05
$\mathrm{d}\chi_{SI}/\mathrm{d}p$	M1	‡	‡	1.02 ± 0.01	0.9 ± 0.15
	BH	1.02 ± 0.03	1.30 ± 0.11	*	*

method used. The functions $1+\chi_Z$, χ_Z/p have the same two leading exponents but the higher-order terms are different. Differentiating χ_Z with respect to p shifts all the exponents by 1 but changes the relative amplitudes. Also, the preprocessing which is carried out in the M1 and M2 methods treats the higher-order terms differently. The wide spread of Δ_1 estimates is showing that a two-exponent fit is not really adequate and that only an effective correction to the scaling exponent is being estimated. Much longer series would be required before the true leading correction to scaling exponent could be singled out. However, the inclusion of Δ_1 in the analysis is important in producing more reliable estimates of the leading exponent [23, 24].

In principle the BH method allows all of the correction to scaling exponents to be estimated by defining a modified function which has a different pole for each correction term. The leading exponent comes from the pole closest to the origin and with only a limited number of terms in the expansion of this function available the poles in the Padé approximants often only give good convergence to this pole. The poles further away from the origin are much less well represented by the approximants and it is sometimes impossible to determine even the first correction. The range of Δ_1 given in the table was determined only from those approximants for which the estimate of ζ_Z fell within the quoted range.

Since χ_{SI} and χ_{BI} have the same critical point and exponent, their amplitude ratio may be found by the method described in the introduction to this section. The results for the two lattices are shown in table 4 together with the estimated correction to scaling exponents.

Table 4. Amplitude ratios for the intersection and union of all paths.

	Square		Triangular	
	Ratio	Δ_1	Ratio	Δ_1
A_{SI}/A_{BI}	1.255 ± 0.007	1.18 ± 0.04	1.335 ± 0.015	1.4 ± 0.1
A_{BU}/A_{SU}	1.188 ± 0.004	1.43 ± 0.04	1.266 ± 0.009	1.52 ± 0.08

The ratios depend on the choice of Δ_1 and the error bars on the ratios correspond to those chosen for this exponent.

The range of the amplitude ratios is determined by equation (6) which gives

$$1 \leqslant \frac{A_{SI}}{A_{BI}} \leqslant 2. \tag{17}$$

The lower limit corresponds to the nodal bonds being connected in a single chain whereas the upper limit would be achieved if the nodal bonds were completely disjoint. The values we find suggest that the nodal bonds form chains of average length 4 for the square lattice and 3 for the triangular lattice.

3.1.2. Union of all paths. The data for the union of all paths is given in table 5.

Bearing in mind our comments for the path intersection data, the estimates are consistent with the exact relation $\zeta_{BU} = \zeta_{SU}$. For the path union the exponent is unknown but there is a conjectured exact value $\zeta_{BU} = \frac{25}{12} = 2.083\,33\ldots$ based on conformal invariance [15] and accurate Monte Carlo estimates 2.196 ± 0.005 [16] and 2.19 ± 0.01 [17]. Our results mostly lie between the theoretical and Monte Carlo values. Those for the bond union are clustered around 2.14 which is midway and the site union results for the square lattice are consistent with this. On the other hand, the site union results for the triangular lattice are scattered around the Monte Carlo value. However, our estimates for the site intersection exponent on the triangular lattice were above the exact value and we therefore favour the square lattice data.

The amplitude ratio A_{BU}/A_{SU} is estimated in table 4. From (5) it follows that

$$1 \leqslant \frac{A_{BU}}{A_{SU}} \leqslant \frac{1}{2}z. \tag{18}$$

The lower limit corresponds to the backbone being a single chain whereas the upper limit would be achieved for compact clusters. The data suggests an effective coordination number for sites in the backbone of about 2.4 for the square lattice and 2.5 for the triangular lattice. We note that by going from the square lattice to the triangular lattice the amplitude ratios for the intersection and union increase by the same factor.

3.2. Union, intersection and length of shortest paths

The data for the union and intersection of shortest paths are given in table 6.

Again the results are clearly consistent with equality of bond and site exponents for both union and intersection. The only surprising feature of the exponents is that in the case of the intersection, the results for the square and triangular lattices are very different. The value for the square lattice corresponds to a fractal dimension of about 1, whereas that for the triangular lattice is 10% higher. This difference is well outside the error to be expected

Table 5. Estimates of the ζ-exponent for the union of all paths.

		Square		Triangular	
		$\gamma_{BI}-\gamma$	Δ_1	$\gamma_{BI}-\gamma$	Δ_1
(a) Bond union					
$1+\chi_{BU}$	M2	2.128 ± 0.003	3.10 ± 0.15	2.14 ± 0.25	1.8 ± 0.2
	M1	2.14 ± 0.01	2.2 ± 0.3	‡	‡
	BH	2.14 ± 0.02	1.9 ± 0.4	2.14 ± 0.03	0.70 ± 0.03
χ_{BU}/p	M2	2.08 ± 0.02	1.60 ± 0.05	2.15 ± 0.02	1.4 ± 0.2
	M1	‡	‡	2.15 ± 0.10	1.2 ± 0.4
	BH	2.12 ± 0.05	1.47 ± 0.15	2.14 ± 0.01	1.87 ± 0.12
$\mathrm{d}\chi_{BU}/\mathrm{d}p$	M2	2.140 ± 0.005	2.25 ± 0.05	2.20 ± 0.01	1.5 ± 0.1
	M1	2.138 ± 0.020	2.00 ± 0.25	2.16 ± 0.03	1.5 ± 0.2
	BH	2.14 ± 0.02	1.56 ± 0.72	2.14 ± 0.02	0.7 ± 0.1
		$\gamma_{SU}-\gamma$	Δ_1	$\gamma_{SU}-\gamma$	Δ_1
(b) Site union					
$1+\chi_{SU}$	M2	2.13 ± 0.03	1.55 ± 0.08	2.208 ± 0.002	2.00 ± 0.02
	M1	2.13 ± 0.02	2.30 ± 0.25	2.25 ± 0.03	1.5 ± 0.5
	BH	2.18 ± 0.03	0.97 ± 0.04	2.18 ± 0.03	1.13 ± 0.40
χ_{SU}/p	M2	2.08 ± 0.05	1.50 ± 0.15	2.19 ± 0.03	1.45 ± 0.05
	M1	2.09 ± 0.04	1.70 ± 0.15	2.14 ± 0.04	1.5 ± 0.5
	BH	2.14 ± 0.30	1.93 ± 0.53	2.17 ± 0.07	1.53 ± 0.24
$\mathrm{d}\chi_{SU}/\mathrm{d}p$	M2	2.11 ± 0.02	1.80 ± 0.05	2.21 ± 0.01	2.00 ± 0.12
	M1	2.14 ± 0.01	1.5 ± 0.2	2.22 ± 0.02	1.3 ± 0.2
	BH	2.17 ± 0.02	1.05 ± 0.08	2.22 ± 0.02	2.0 ± 0.8

from the spread of results for each lattice taken separately and strongly suggests that this exponent is non-universal.

The amplitude ratios for intersection and union are given in table 7. The same inequalities apply as for the 'all path' ratios but for the shortest paths the ratios are closer to the bottom of the ranges. This is particularly noticeable in the case of the path intersection which means that the nodal bonds of the elastic backbone are partitioned into longer chains, there being fewer parallel paths in the backbone. Based on the data, the average chain length for large clusters is about 13.

The exponent estimates for the shortest path length are given in table 8. As for the nodal bonds there is a noticeable difference between the exponents for the square and triangular lattices but this time it is only 4% and may not be significant. The values lie between those for the union and intersection in agreement with the inequalities of equation (3) which appear to be strict. The exponent of the square lattice is the same as that for the nodal bonds on the triangular lattice.

3.3. Union, intersection and length of longest self-avoiding paths

The main feature which is apparent from the longest path data in table 9 is that the exponents for the union and intersection are the same and independent of the lattice.

In addition to the exact equality of bond and site exponents we further conjecture that

Table 6. Estimates of the ζ-exponent from the 'shortest path' series.

		Square		Triangular	
		$\gamma_{BU}-\gamma$	Δ_1	$\gamma_{BU}-\gamma$	Δ_1
(a) Bond union					
$1+\chi_{BU}$	M2	1.601 ± 0.004	1.25 ± 0.05	1.59 ± 0.01	1.75 ± 0.30
	M1	1.59 ± 0.02	1.14 ± 0.04	1.58 ± 0.01	1.40 ± 0.15
	BH	1.58 ± 0.02	1.6 ± 0.4	1.57 ± 0.02	1.44 ± 0.18
χ_{BU}/p	M2	1.58 ± 0.03	1.15 ± 0.06	1.54 ± 0.02	1.1 ± 0.2
	M1	1.58 ± 0.01	1.1 ± 0.2	1.59 ± 0.02	1.2 ± 0.3
	BH	1.55 ± 0.02	1.35 ± 0.06	1.58 ± 0.04	1.04 ± 0.13
$d\chi_{BU}/dp$	M2	1.60 ± 0.01	1.15 ± 0.03	1.58 ± 0.02	1.50 ± 0.15
	M1	1.58 ± 0.02	1.5 ± 0.2	1.59 ± 0.01	1.6 ± 0.2
	BH	1.58 ± 0.01	1.3 ± 0.1	1.57 ± 0.05	1.45 ± 0.22
		$\gamma_{SU}-\gamma$	Δ_1	$\gamma_{SU}-\gamma$	Δ_1
(b) Site union					
$1+\chi_{SU}$	M2	1.559 ± 0.003	2.20 ± 0.05	1.58 ± 0.01	1.80 ± 0.05
	M1	1.57 ± 0.01	2.05 ± 0.20	1.57 ± 0.02	1.7 ± 0.2
	BH	1.56 ± 0.01	1.9 ± 0.4	1.57 ± 0.01	1.64 ± 0.21
χ_{SU}/p	M2	1.54 ± 0.02	1.20 ± 0.15	1.55 ± 0.02	0.9 ± 0.2
	M1	1.56 ± 0.01	2.5 ± 0.2	1.56 ± 0.03	1.1 ± 0.3
	BH	1.56 ± 0.03	0.95 ± 0.24	1.53 ± 0.03	1.1 ± 0.2
$d\chi_{SU}/dp$	M2	1.561 ± 0.008	2.50 ± 0.04	1.57 ± 0.01	1.60 ± 0.04
	M1	1.56 ± 0.01	2.5 ± 0.2	1.56 ± 0.02	1.6 ± 0.2
	BH	1.56 ± 0.01	2.40 ± 0.07	1.57 ± 0.01	1.70 ± 0.15
		$\gamma_{BI}-\gamma$	Δ_1	$\gamma_{BI}-\gamma$	Δ_1
(c) Bond intersection					
$1+\chi_{BI}$	M2	1.337 ± 0.003	1.95 ± 0.02	1.49 ± 0.01	1.95 ± 0.17
	M1	1.33 ± 0.02	2.2 ± 0.3	1.47 ± 0.2	2.0 ± 0.2
	BH	1.33 ± 0.03	1.3 ± 0.3	1.47 ± 0.20	2.1 ± 0.4
χ_{BI}/p	M2	1.36 ± 0.03	†	‡	‡
	M1	1.38 ± 0.02	1.10 ± 0.15	‡	‡
	BH	1.34 ± 0.02	1.1 ± 0.2	1.50 ± 0.02	0.79 ± 0.30
$d\chi_{BI}/dp$	M2	1.34 ± 0.02	2.20 ± 0.11	1.466 ± 0.007	1.80 ± 0.08
	M1	1.34 ± 0.01	2.2 ± 0.3	1.47 ± 0.20	2.0 ± 0.5
	BH	1.36 ± 0.02	2.40 ± 0.08	1.46 ± 0.02	1.71 ± 0.04
		$\gamma_{SI}-\gamma$	Δ_1	$\gamma_{SI}-\gamma$	Δ_1
(d) Site intersection					
$1+\chi_{SI}$	M2	1.37 ± 0.01	1.90 ± 0.06	1.47 ± 0.01	1.75 ± 0.05
	M1	1.39 ± 0.02	2.25 ± 0.03	1.47 ± 0.01	1.80 ± 0.02
	BH	1.39 ± 0.01	2.80 ± 0.04	1.49 ± 0.02	1.95 ± 0.33
χ_{SI}/p	M2	1.39 ± 0.03	0.90 ± 0.05	‡	‡
	M1	1.37 ± 0.02	2.02 ± 0.07	1.47 ± 0.04	1.5 ± 0.5
	BH	1.36 ± 0.03	1.17 ± 0.04	1.49 ± 0.05	0.8 ± 0.3
$d\chi_{SI}/dp$	M2	1.37 ± 0.01	1.85 ± 0.05	1.47 ± 0.01	1.7 ± 0.2
	M1	1.37 ± 0.20	2.07 ± 0.08	1.49 ± 0.02	2.0 ± 0.03
	BH	1.38 ± 0.02	2.27 ± 0.09	1.47 ± 0.02	1.7 ± 0.2

Table 7. Amplitude ratios for the intersection and union of shortest paths.

	Square		Triangular	
	Ratio	Δ_1	Ratio	Δ_1
A_{SI}/A_{BI}	1.073 ± 0.009	1.6 ± 0.2	1.052 ± 0.002	1.00 ± 0.04
A_{BU}/A_{SU}	1.082 ± 0.005	1.4 ± 0.1	1.054 ± 0.004	1.0 ± 0.1

Table 8. Estimates of the ζ-exponent from the 'shortest path length' series.

		Square		Triangular	
		$\gamma_{min} - \gamma$	Δ_1	$\gamma_{min} - \gamma$	Δ_1
$1 + \chi_{min}$	M2	1.483 ± 0.003	2.25 ± 0.05	1.53 ± 0.01	1.8 ± 0.2
	M1	1.48 ± 0.02	1.20 ± 0.04	1.53 ± 0.01	1.85 ± 0.05
	BH	1.48 ± 0.01	2.22 ± 0.24	1.53 ± 0.02	1.84 ± 0.19
χ_{min}/p	M2	1.47 ± 0.01	2.00 ± 0.02	1.54 ± 0.03	0.90 ± 0.05
	M1	1.47 ± 0.03	1.00 ± 0.05	1.53 ± 0.03	0.9 ± 0.1
	BH	1.48 ± 0.02	2.8 ± 0.6	1.46 ± 0.04	1.21 ± 0.25
$\mathrm{d}\chi_{min}/\mathrm{d}p$	M2	1.47 ± 0.01	2.20 ± 0.05	1.53 ± 0.01	1.80 ± 0.05
	M1	1.47 ± 0.01	2.20 ± 0.03	1.53 ± 0.01	1.80 ± 0.05
	BH	1.48 ± 0.02	2.21 ± 0.28	1.53 ± 0.02	1.74 ± 0.11

$\zeta_{BI} = \zeta_{BU}$. Equation (4) then implies that the longest path length exponent ζ_{max} also has the same value. This is borne out by the path length data in table 10. The equality of these exponents would be explained if the longest self-avoiding path between two points on a typical cluster near the critical point were unique.

If, further, the longest path visited nearly all of the sites in the backbone then the path length would be essentially equal to the number of sites and we would get $\zeta_{max} = \nu D_B$. Comparing the data in table 5 with that in tables 9 and 10 we see that the estimates of ζ_{max} are about 3% lower and are closer to the rational value $\frac{25}{12}$ which is the field theory conjecture [15] for νD_B.

Assuming that $\zeta_{BI} = \zeta_{SI} = \zeta_{max} = \zeta_{SU} = \zeta_{BU}$ we have determined the ratios of the amplitudes corresponding to the last four exponents to that of the first and find that they form an increasing sequence as expected. The data is given in table 11. The bond-to-site ratios for both intersection and union are very close to 1 which is consistent with them typically being a single longest path.

4. Discussion

Our results for the longest self-avoiding paths suggest that the union, intersection and length exponents are all equal and the common value of ζ found is close to $\frac{25}{12}$ which is the field theory value for the union of all paths [14, 15]. This would be explained if the longest path on a typical backbone visited nearly all of the backbone sites. Our estimate of ζ_{BU} for the union of all paths is about 3% higher than the field-theory value but in view of the discussion in section 3.1.1 this difference may not be significant. However, the most recent Monte Carlo results [16] give a value which is 6% higher than the field theory.

Table 9. Estimates of the ζ-exponent from the 'longest path' series.

		Square		Triangular	
		$\gamma_{BU}-\gamma$	Δ_1	$\gamma_{BU}-\gamma$	Δ_1
(a) Bond union					
	M2	2.06 ± 0.01	1.4 ± 0.1	2.11 ± 0.01	2.40 ± 0.05
$1+\chi_{BU}$	M1	2.08 ± 0.02	1.35 ± 0.05	2.12 ± 0.01	2.6 ± 0.2
	BH	2.05 ± 0.04	*	2.12 ± 0.03	2.63 ± 0.80
	M2	2.06 ± 0.02	1.15 ± 0.05	2.11 ± 0.01	1.05 ± 0.05
χ_{BU}/p	M1	2.05 ± 0.02	1.20 ± 0.01	2.11 ± 0.03	1.05 ± 0.15
	BH	2.04 ± 0.03	1.33 ± 0.17	2.08 ± 0.03	1.47 ± 0.06
	M2	2.05 ± 0.02	1.35 ± 0.05	2.11 ± 0.01	2.50 ± 0.15
$\mathrm{d}\chi_{BU}/\mathrm{d}p$	M1	2.06 ± 0.01	1.20 ± 0.01	2.116 ± 0.005	2.45 ± 0.07
	BH	2.03 ± 0.04	1.52 ± 0.50	2.12 ± 0.02	2.72 ± 0.09
		$\gamma_{SU}-\gamma$	Δ_1	$\gamma_{SU}-\gamma$	Δ_1
(b) Site union					
	M2	2.01 ± 0.01	1.70 ± 0.02	2.08 ± 0.03	2.40 ± 0.05
$1+\chi_{SU}$	M1	2.02 ± 0.04	1.5 ± 0.3	2.095 ± 0.004	2.30 ± 0.15
	BH	1.94 ± 0.06	*	2.10 ± 0.02	2.30 ± 0.01
	M2	2.01 ± 0.01	1.2 ± 0.1	2.09 ± 0.02	1.05 ± 0.04
χ_{SU}/p	M1	2.02 ± 0.30	1.2 ± 0.1	‡	‡
	BH	2.02 ± 0.02	1.37 ± 0.08	2.05 ± 0.20	1.46 ± 0.13
	M2	2.028 ± 0.005	1.35 ± 0.05	2.096 ± 0.002	2.60 ± 0.08
$\mathrm{d}\chi_{SU}/\mathrm{d}p$	M1	2.02 ± 0.02	1.2 ± 0.5	2.096 ± 0.005	2.45 ± 0.17
	BH	2.02 ± 0.03	1.96 ± 0.15	2.10 ± 0.01	2.30 ± 0.01
		$\gamma_{BI}-\gamma$	Δ_1	$\gamma_{BI}-\gamma$	Δ_1
(c) Bond intersection					
	M2	2.00 ± 0.02	1.25 ± 0.05	2.045 ± 0.006	2.55 ± 0.08
$1+\chi_{BI}$	M1	2.02 ± 0.03	1.20 ± 0.15	2.059 ± 0.002	2.5 ± 0.2
	BH	2.05 ± 0.05	1.25 ± 0.43	2.06 ± 0.02	2.81 ± 0.40
	M2	1.99 ± 0.02	1.20 ± 0.05	2.05 ± 0.01	1.2 ± 0.1
χ_{BI}/p	M1	‡	‡	‡	‡
	BH	1.98 ± 0.07	1.4 ± 0.3	2.05 ± 0.30	1.35 ± 0.02
	M2	1.99 ± 0.03	1.90 ± 0.01	2.052 ± 0.005	3.00 ± 0.18
$\mathrm{d}\chi_{BI}/\mathrm{d}p$	M1	‡	‡	2.057 ± 0.002	2.70 ± 0.04
	BH	2.04 ± 0.04	*	2.06 ± 0.02	2.44 ± 0.04
		$\gamma_{SI}-\gamma$	Δ_1	$\gamma_{SI}-\gamma$	Δ_1
(d) Site intersection					
	M2	2.02 ± 0.02	1.5 ± 0.05	2.07 ± 0.02	2.6 ± 0.03
$1+\chi_{SI}$	M1	2.04 ± 0.03	1.3 ± 0.06	2.07 ± 0.03	2.7 ± 0.06
	BH	1.99 ± 0.04	2.21 ± 0.30	2.08 ± 0.02	2.6 ± 0.3
	M2	2.02 ± 0.01	1.20 ± 0.05	‡	‡
χ_{SI}/p	M1	2.02 ± 0.02	1.20 ± 0.05	2.06 ± 0.04	1.5 ± 0.7
	BH	1.98 ± 0.07	1.48 ± 0.12	2.05 ± 0.02	1.43 ± 0.09
	M2	2.04 ± 0.02	1.50 ± 0.06	2.075 ± 0.002	3.00 ± 0.05
$\mathrm{d}\chi_{SI}/\mathrm{d}p$	M1	2.04 ± 0.01	1.40 ± 0.05	2.06 ± 0.02	1.15 ± 0.05
	BH	1.84 ± 0.07	1.00 ± 0.03	2.08 ± 0.02	3.06 ± 0.08

Table 10. Estimates of the ζ-exponent from the 'longest path length' series.

		Square		Triangular	
		$\gamma_{\max}-\gamma$	Δ_1	$\gamma_{\max}-\gamma$	Δ_1
$1+\chi_{\max}$	M2	1.97 ± 0.02	2.00 ± 0.2	2.08 ± 0.02	2.30 ± 0.05
	M1	‡	‡	2.08 ± 0.02	1.40 ± 0.05
	BH	2.00 ± 0.09	2.22 ± 0.26	2.09 ± 0.01	2.85 ± 0.20
$\chi_{\max}/p$	M2	2.00 ± 0.02	1.50 ± 0.15	2.09 ± 0.01	1.25 ± 0.04
	M1	‡	‡	2.07 ± 0.02	1.1 ± 0.2
	BH	1.97 ± 0.08	1.5 ± 0.2	2.05 ± 0.02	1.36 ± 0.07
$\mathrm{d}\chi_{\max}/\mathrm{d}p$	M2	1.98 ± 0.01	2.1 ± 0.2	2.09 ± 0.02	1.4 ± 0.1
	M1	‡	‡	2.08 ± 0.02	2.4 ± 0.2
	BH	1.94 ± 0.12	*	2.09 ± 0.01	2.86 ± 0.20

Table 11. Amplitude ratios for the longest paths.

	Square		Triangular	
	Ratio	Δ_1	Ratio	Δ_1
A_{SI}/A_{BI}	1.046 ± 0.006	2.30 ± 0.05	1.038 ± 0.003	2.0 ± 0.1
$A_{\max}/A_{BI}$	1.13 ± 0.01	1.90 ± 0.05	1.058 ± 0.005	2.3 ± 0.1
A_{SU}/A_{BI}	1.189 ± 0.003	1.94 ± 0.02	1.079 ± 0.004	2.4 ± 0.1
A_{BU}/A_{BI}	1.238 ± 0.005	2.00 ± 0.05	1.116 ± 0.007	2.2 ± 0.1
A_{BU}/A_{SU}	1.045 ± 0.002	2.10 ± 0.05	1.032 ± 0.003	2.15 ± 0.05

The exponents for the union, intersection and length of the shortest paths are found to be different. The values are shown in table 1 and in the case of the union, the square and triangular lattice estimates for both bond and site unions are consistent with a common value as expected. On the other hand, the exponent estimates for bond and site intersections agree well with the theoretical equality but they appear to be lattice dependent. As can be seen from table 6, the difference between the lattices is consistently at the 10% level which is much greater than any deviations from the theoretical value found for the intersection of all paths in section 3.1.1. A difference of smaller magnitude but in the same direction also occurs for the shortest path length but this is of smaller magnitude. It is difficult to attribute the difference in the case of the intersection to errors arising from short series effects in view of the general overall consistency of our results for other properties. This apparent violation of universality deserves further investigation. We know of no field theory formulations from which universality would follow in the case of shortest paths.

Acknowledgments

We would like to thank Dr Joan Adler and colleagues for clarifying the difference between their definition of backbone and that of Hong and Stanley [5]. This work was carried out in part during the tenure, by JWE, of a Senior Visiting Fellowship at the Centre for Chemical Physics, University of Western Ontario. FMB thanks the University of Brunei Darussalam for research leave and Royal Holloway, University of London for their hospitality which together enabled the completion of this work.

Appendix

	Square lattice: all paths			
n	Bond union	Bond intersection	Site union	Site intersection
0	0	0	0	0
1	4	4	4	4
2	24	24	24	24
3	108	108	108	108
4	400	352	388	364
5	1 372	1 180	1 324	1 228
6	4 296	3 168	4 044	3 420
7	13 020	9 744	12 256	10 448
8	37 072	22 624	34 084	25 412
9	104 052	68 472	96 128	74 960
10	278 456	143 120	251 840	165 656
11	742 236	432 828	675 828	481 300
12	1 899 104	836 448	1 697 124	989 708
13	4 881 536	2 569 060	4 397 412	2 885 252
14	12 068 880	4 455 248	10 672 092	5 419 464
15	30 189 440	15 201 360	27 026 472	16 983 064
16	72 698 936	20 462 592	63 591 408	26 454 948
17	177 995 708	91 165 560	158 861 544	99 938 008
18	419 317 760	79 539 912	363 158 568	115 407 424

	Square lattice: shortest paths				
n	Path length	Bond union	Bond intersection	Site union	Site intersection
0	0	0	0	0	0
1	4	4	4	4	4
2	24	24	24	24	24
3	108	108	108	108	108
4	368	376	360	372	364
5	1 244	1 276	1 212	1 260	1 228
6	3 532	3 696	3 368	3 620	3 444
7	10 776	11 276	10 284	11 040	10 512
8	27 084	29 016	25 184	28 164	26 004
9	79 112	84 308	74 092	81 952	76 272
10	183 132	200 176	166 696	192 900	173 396
11	521 604	565 196	480 140	546 020	497 244
12	1 137 460	1 265 256	1 016 552	1 212 108	1 063 448
13	3 209 276	3 525 744	2 912 768	3 389 284	3 030 436
14	6 535 720	7 406 344	5 724 136	7 053 072	6 025 256
15	19 165 128	21 216 204	17 277 044	20 341 400	18 004 016
16	34 442 416	40 040 840	29 272 560	37 828 492	31 114 436
17	112 760 768	124 990 396	101 708 092	119 770 008	105 890 168
18	168 817 788	202 649 304	137 870 000	189 660 356	148 430 860

n	Square lattice: longest paths Path length	Bond union	Bond intersection	Site union	Site intersection
0	0	0	0	0	0
1	4	4	4	4	4
2	24	24	24	24	24
3	108	108	108	108	108
4	384	392	376	388	380
5	1 308	1 340	1 276	1 324	1 292
6	3 932	4 096	3 768	4 020	3 844
7	11 936	12 408	11 456	12 184	11 680
8	32 460	34 288	30 600	33 476	31 412
9	92 224	96 856	87 416	94 720	89 552
10	235 260	250 976	219 064	244 100	225 916
11	638 892	676 772	599 148	659 476	616 412
12	1 558 364	1 674 672	1 437 640	1 623 764	1 488 124
13	4 104 932	4 378 044	3 816 924	4 253 148	3 941 188
14	9 622 632	10 415 752	8 796 448	10 069 576	9 138 288
15	25 069 504	26 825 116	23 210 192	26 008 048	24 019 288
16	56 044 256	61 184 832	50 687 960	58 961 380	52 877 396
17	147 335 064	157 753 572	136 235 152	152 784 728	141 137 424
18	312 180 572	343 782 624	279 293 256	330 228 708	292 596 900

n	Triangular lattice: all paths Bond union	Bond intersection	Site union	Site intersection
0	0	0	0	0
1	6	6	6	6
2	60	60	60	60
3	414	378	402	390
4	2 376	1 944	2 244	2 076
5	12 168	8 940	11 232	9 840
6	57 540	38 124	52 116	43 056
7	255 966	152 628	228 012	176 544
8	1 086 252	591 360	955 146	695 760
9	4 438 602	2 207 736	3 858 900	2 639 028
10	17 575 092	8 017 800	15 131 934	9 716 034
11	67 805 790	28 746 630	57 925 650	35 137 170
12	255 863 892	100 081 080	216 969 420	123 761 760
13	947 159 934	346 964 514	798 427 518	431 567 298
14	3449 198 736	1181 211 108	2891 756 988	1480 037 712

	Triangular lattice: shortest paths				
n	Path length	Bond union	Bond intersection	Site union	Site intersection
0	0	0	0	0	0
1	6	6	6	6	6
2	60	60	60	60	60
3	390	390	390	390	390
4	2 088	2 112	2 064	2 100	2 076
5	9 978	10 134	9 822	10 056	9 900
6	44 166	45 312	43 020	44 784	43 548
7	183 690	189 198	178 230	186 624	180 756
8	735 420	764 772	706 404	751 962	718 902
9	2 837 544	2 957 508	2 719 992	2 903 700	2 771 496
10	10 644 540	11 203 044	10 098 912	10 968 870	10 321 794
11	39 171 978	41 232 996	37 173 900	40 333 938	38 015 034
12	140 741 538	149 679 696	132 102 972	146 055 348	135 475 512
13	499 516 200	530 083 518	470 166 606	516 970 254	482 193 426
14	1745 462 190	1873 971 588	1622 353 824	1823 575 512	1668 350 010

	Triangular lattice: longest paths				
n	Path length	Bond union	Bond intersection	Site union	Site intersection
0	0	0	0	0	0
1	6	6	6	6	6
2	60	60	60	60	60
3	402	402	402	402	402
4	2 232	2 256	2 208	2 244	2 220
5	11 094	11 262	10 926	11 172	11 016
6	51 042	52 164	49 920	51 600	50 484
7	221 244	226 776	215 652	223 896	218 544
8	918 570	945 864	890 940	932 052	904 824
9	3 678 150	3 796 458	3 557 406	3 734 796	3 619 680
10	14 297 466	14 808 996	13 776 012	14 545 026	14 042 238
11	54 312 144	56 363 856	52 209 924	55 282 086	53 304 510
12	201 809 142	210 022 572	193 404 624	205 699 062	197 770 482
13	737 550 168	768 650 016	705 647 112	751 980 714	722 504 970
14	2653 237 548	2772 100 584	2531 314 512	2708 706 114	2595 380 442

References

[1] Skal A S and Shklovskii B I 1975 *Sov. Phys. Semicond.* **8** 1029–32
[2] de Gennes P G 1976 *J. Physique Lett.* **37** L1–2
[3] Pike R and Stanley H E 1981 *J. Phys. A: Math. Gen.* **14** L169
[4] Harris A B and Fisch R 1977 *Phys. Rev. Lett.* **38** 796–9
[5] Hong D C and Stanley H E 1983 *J. Phys. A: Math. Gen.* **16** L475–81
[6] Coniglio A 1981 *Phys. Rev. Lett.* **46** 250–3
[7] Herrmann H J, Hong D C and Stanley H E 1984 *J. Phys. A: Math. Gen.* **17** L261–6
[8] Brak R, Essam J W and Sykes M F 1989 *J. Phys. A: Math. Gen.* **21** 3361–9
[9] Herrmann H J and Stanley H E 1988 *J. Phys. A: Math. Gen.* **21** L829–33
[10] Grassberger P 1985 *J. Phys. A: Math. Gen.* **18** L215–19
[11] den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857–68
[12] Coniglio A 1982 *J. Phys. A: Math. Gen.* **15** 3829–44

[13] Adler J, Aharony A, Blumenfeld R, Harris A B and Meir Y 1993 *Phys. Rev.* B **47** 5770–82
[14] Larsson 1987 *J. Phys. A: Math. Gen.* **20** L291–7
[15] Saleur H 1992 *Nucl. Phys.* B **382** 486–531
[16] Grassberger P 1992 *J. Phys. A: Math. Gen.* **25** 5475–84
[17] Rintoul M D and Nakanishi H 1992 *J. Phys. A: Math. Gen.* **25** L945–8
[18] Herrmann H J and Stanley H E 1984 *Phys. Rev. Lett.* **53** 1121–4
[19] Laidlaw D, MacKay G and Jan N 1987 *J. Stat. Phys.* **46** L507–15
[20] Arbabi S and Sahimi M 1993 *Phys. Rev.* B **47** 695–702
[21] Bhatti F M and Essam J W 1988 *Disc. Appl. Math.* **19** 85–112
[22] Conway A G and Guttmann A J 1995 *J. Phys. A: Math. Gen.* **28** 891–904
[23] Adler J, Moshe M and Privman V 1982 *Phys. Rev.* B **26** 1411–15
[24] Adler J, Meir Y, Aharony A, Harris A B and Klein L 1990 *J. Stat. Phys.* **58** 511–38
[25] Baker G A and Hunter D L 1973 *Phys. Rev.* B **7** 3373–92
[26] Essam J W, Lookman T and De'Bell K 1996 *J. Phys. A: Math. Gen.* **29** L143–50